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Some inequalities for warped products in locally conformal almost cosymplectic manifolds

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Abstract. In this article, we investigate the inequality between the warping function of a warped product submanifold isometrically immersed in locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature and the squared mean curvature. Furthermore, some applications are derived.

Keywords: Warped product, mean curvature, minimal immersion, inequality, totally real submanifold, locally conformal almost cosymplectic manifold.

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Introduction

Let M_1 and M_2 be Riemannian manifolds of positive dimension n_1 and n_2, equipped with Riemannian metrics g_1 and g_2, respectively. Let f be a positive function on M_1. The warped product $M_1 \times_f M_2$ is defined to be the product manifold $M_1 \times M_2$ with the warped metric: $g = g_1 + f^2 g_2$ (see [3]).

It is well-known that the notion of warped products plays some important role in differential geometry as well as in physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [3].

Let $x : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold $\tilde{M}(c)$ with constant sectional curvature c. We denote by h the second fundamental form of x and $H_i = \frac{1}{n_i}\text{trace } h_i$, where trace h_i is the trace of h restricted to M_i. We call H_i ($i = 1, 2$) the partial mean curvature vectors. The immersion x is said to be *mixed totally geodesic* if $h(X,Z)=0$ for any vector fields X and Z tangent to M_1 and M_2, respectively.

Recently, in [4] B.-Y. Chen established the following sharp relationship between the warping function f of a warped product $M_1\times_f M_2$ isometrically immersed in a real space form $\tilde{M}(c)$ and the squared mean curvature $||H||^2$:

1 Theorem ([4]). *Let $x: M_1\times_f M_2\longrightarrow \tilde{M}(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then, we have*

$$\frac{\Delta f}{f}\leq\frac{(n_1+n_2)^2}{4n_2}||H||^2+n_1c, \tag{1}$$

where Δ is the Laplacian operator of M_1.

As an immediate application, he obtained necessary conditions for a warped product to admit a minimal isometric immersion in a Euclidean space or in a real space form.

On the other hand, for the above related researches B.-Y. Chen investigated the inequality (1) of a warped product submanifold into complex hyperbolic space [7] and complex projective space form ([5]). Also, K. Matsumoto and I. Mihai ([9]) studied the inequality (1) of a warped product submanifold into Sasakian space form of constant φ-sectional curvature, and the first author and Y. H. Kim ([7]) studied the inequality (1) of a totally real warped product submanifold into locally conformal Kaehler space form.

In this paper, we prove a similar inequality for warped product submanifolds of locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c.

1 Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold with almost contact structure (φ,ξ,η), i.e., a global vector field ξ, a $(1,1)$ tensor field φ and a 1-form η on $\tilde{M}$ such that $\varphi^2X=-X+\eta(X)\xi,\quad \eta(\xi)=1$ for any vector field X on $\tilde{M}$. We consider a product manifold $\tilde{M}\times\mathbb{R}$, where $\mathbb{R}$ denote a real line. Then a vector field on $\tilde{M}\times\mathbb{R}$ is given by $(X,\lambda\frac{d}{dt})$, where X is a vector field tangent to $\tilde{M}$, t the coordinate of $\mathbb{R}$ and λ a function on $\tilde{M}\times\mathbb{R}$. We define a linear map J on the tangent space of $\tilde{M}\times\mathbb{R}$ by $J(X,\lambda\frac{d}{dt})=(\varphi X-\lambda\xi,\eta(X)\frac{d}{dt})$. Then we have $J^2=-I$ and hence J is an almost complex structure on $\tilde{M}\times\mathbb{R}$. The manifold $\tilde{M}$ is said to be *normal* ([1]) if the almost complex structure J is integrable (i.e., J arises from a complex structure on $\tilde{M}\times\mathbb{R}$). The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi,\varphi]+2d\eta\otimes\xi$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ. Let g be a Riemannian metric on $\tilde{M}$

compatible with (φ, ξ, η), that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X and Y tangent to $\tilde{M}$. Thus, the manifold $\tilde{M}$ is almost contact metric, and (φ, ξ, η, g) is its almost contact metric structure. Clearly, we have $\eta(X) = g(X, \xi)$ for any vector field X tangent to $\tilde{M}$. Let Φ denote the fundamental 2-form of $\tilde{M}$ defined by $\Phi(X, Y) = g(\varphi X, Y)$ for any vector fields X and Y tangent to $\tilde{M}$. The manifold $\tilde{M}$ is said to be *almost cosymplectic* if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$, where d is the operator of exterior differentiation. If $\tilde{M}$ is almost cosymplectic and normal, then it is called *cosymplectic* ([1]). It is well know that the almost contact metric manifold is cosymplectic if and only if $\tilde{\nabla}\varphi$ vanishes identically, where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. An almost contact metric manifold $\tilde{M}$ is called a *locally conformal almost cosymplectic manifold* ([12]) if there exists a 1-form ω such that $d\Phi = 2\omega \wedge \Phi$, $d\eta = \omega \wedge \eta$ and $d\omega = 0$.

A necessary and sufficient condition for a structure to be normal locally conformal almost cosympletic is ([10])

$$(\tilde{\nabla}_X \varphi)Y = u(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{2}$$

where $\omega = u\eta$. From formula (2) it follows that $\tilde{\nabla}_X \xi = u(X - \eta(X)\xi)$.

A plane section σ in $T_p\tilde{M}$ of an almost contact structure manifold $\tilde{M}$ is called a *φ-section* if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. $\tilde{M}$ is of pointwise constant φ-sectional curvature if at each point $p \in \tilde{M}$, the section curvature $\tilde{K}(\sigma)$ does not depend on the choice of the φ-section σ of $T_p\tilde{M}$, and in this case for $p \in \tilde{M}$ and for any φ-section σ of $T_p\tilde{M}$, the function c defined by $c(p) = \tilde{K}(\sigma)$ is called the *φ-sectional curvature* of $\tilde{M}$. A locally conformal almost cosymplectic manifolds $\tilde{M}$ of dimension ≥ 5 is of pointwise constant φ-sectional curvature if and only if its curvature tensor $\tilde{R}$ is of the form ([12])

$$\begin{aligned}
\tilde{R}(X, Y, W, Z) &= \frac{c - 3u^2}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\
&+ \frac{c + u^2}{4}\{g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) \\
&- 2g(X, \varphi Y)g(Z, \varphi W)\} \\
&- \left(\frac{c + u^2}{4} + u'\right)\{g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W) \\
&+ g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)\} \\
&+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),
\end{aligned} \tag{3}$$

where u is the function such that $\omega = u\eta$, $u' = \xi u$, and c is the pointwise constant φ-sectional curvature of $\tilde{M}$.

Let M be an n-dimensional submanifold immersed in a locally conformal almost cosymplectic manifold $\tilde{M}$. Let ∇ be the induced Levi-Civita connection of M. Then the Gauss and Weingarten formulas given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\tilde{\nabla}_X V = -A_V X + D_X V$$

for vector fields X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, D the normal connection and A_V the shape operator in the direction of V. The second fundamental form and the shape operator are related by

$$g(h(X,Y),V) = g(A_V X, Y).$$

We also use g for the induced Riemannian metric on M as well as the locally conformal almost cosymplectic manifold $\tilde{M}$.

For any vector X tangent to M we put $\varphi X = PX + FX$, where PX and FX are the tangential and the normal components of φX, respectively. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of M, we define the squared norm of P by

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j) \tag{4}$$

and the mean curvature vector $H(p)$ at $p \in M$ is given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$.

We put

$$h_{ij}^r = g(h(e_i,e_j),e_r) \quad \text{and} \quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j))$$

where $\{e_{n+1}, \ldots, e_{2m+1}\}$ is an orthonormal basis of $T_p^{\perp}M$ and $r = n+1, \ldots, 2m+1$.

A submanifold M is *totally geodesic* in $\tilde{M}$ if $h = 0$, and *minimal* if $H = 0$.

On the other hand, M is said to be a *totally real submanifold* if P is identically zero, that is, $\varphi X \in T_p^{\perp}M$ for any $X \in T_pM, p \in M$.

For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$. For any orthornormal basis $e_1, \ldots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined by to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{5}$$

2 Some inequality for warped product submanifolds

We give the following lemma for later use.

2 Lemma ([2]). *Let* $a_1, \ldots, a_n, a_{n+1}$ *be* $n+1$ $(n \geq 2)$ *real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + a_{n+1}\right).$$

Then, $2a_1a_2 \geq a_{n+1}$*, with the equality holding if and only if* $a_1 + a_2 = a_3 = \cdots = a_n$.

We investigate warped product submanifolds tangent to the structure vector field ξ in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$.

3 Theorem. *Let* $x : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ *be an isometric immersion of an* n*-dimensional warped product into a* $(2m+1)$*-dimensional locally conformal almost cosymplectic manifold of pointwise constant* φ*-sectional curvature* c *whose structure vector field* ξ *is tangent to* M_1*. Then, we have*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}||H||^2 + \frac{c-3u^2}{4}n_1 - \left(\frac{c+u^2}{4} + u'\right) + \frac{3(c+u^2)}{4}, \tag{6}$$

where $n_i = dim M_i, i = 1, 2,$ *and* Δ *is the Laplacian operator of* M_1*.*

Proof. Let $M_1 \times_f M_2$ be a warped product submanifold of a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ with pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. Since $M_1 \times_f M_2$ is a warped product, it is easily seen that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{7}$$

for any vector fields X, Z tangent to M_1, M_2, respectively. If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{8}$$

We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m+1}\}$ such that $e_1, \ldots, e_{n_1} = \xi$ are tangent to M_1, $e_{n_1+1}, \ldots, e_n$ are tangent to M_2 and e_{n+1} is parallel to H. Then, using (8) we obtain

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{9}$$

for each $s \in \{n_1+1, \dots, n\}$.

From the equation of Gauss, we obtain

$$2\tau = \frac{c-3u^2}{4}n(n-1)+\frac{3(c+u^2)}{4}||P||^2-\left(\frac{c+u^2}{4}+u'\right)(2n-2)+n^2||H||^2-||h||^2. \tag{10}$$

We denote

$$\delta = 2\tau - \frac{c-3u^2}{4}n(n-1)-\frac{3(c+u^2)}{4}||P||^2+\left(\frac{c+u^2}{4}+u'\right)(2n-2)-\frac{n^2}{2}||H||^2. \tag{11}$$

Substituting (10) in (11), we have

$$n^2||H||^2 = 2(\delta + ||h||^2). \tag{12}$$

With respect to the above orthonormal basis, (12) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left(\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right),$$

which implies

$$\begin{aligned}\left(\sum_{i=1}^{3} a_i\right)^2 = 2\Bigg\{&\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\neq j\le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\ &- \sum_{2\le j\neq k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\neq t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\Bigg\},\end{aligned} \tag{13}$$

where $a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$.

Applying Lemma 1 to (13) yields

$$\begin{aligned}&\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1} \\ &\ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n}(h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^r)^2,\end{aligned} \tag{14}$$

with equality holding if and only if we have

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{15}$$

Using the Gauss equation, we have from (9)

$$\begin{aligned}
n_2\frac{\Delta f}{f} &= \tau - \sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t)\\
&= \tau - \frac{c-3u^2}{8}n_1(n_1-1) - \sum_{1\le j<k\le n_1} g^2(Pe_j,e_k)\frac{3(c+u^2)}{4}\\
&\quad + \left(\frac{c+u^2}{4}+u'\right)(n_1-1) - \sum_{r=n+1}^{2m+1}\sum_{1\le j<k\le n_1}(h^r_{jj}h^r_{kk}-(h^r_{jk})^2) \qquad (16)\\
&\quad - \frac{c-3u^2}{8}n_2(n_2-1) - \sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t)\frac{3(c+u^2)}{4}\\
&\quad - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le s<t\le n}(h^r_{ss}h^r_{tt}-(h^r_{st})^2).
\end{aligned}$$

Combining (14) and(16) and taking account of (9), we have

$$\begin{aligned}
n_2\frac{\Delta f}{f} \le& \tau - \frac{c-3u^2}{8}n(n-1) + \frac{c-3u^2}{4}n_1n_2 - \frac{\delta}{2} + \left(\frac{c+u^2}{4}+u'\right)(n_1-1)\\
&- \sum_{1\le j<k\le n_1} g^2(Pe_j,e_k)\frac{3(c+u^2)}{4} - \sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t)\frac{3(c+u^2)}{4}.
\end{aligned} \tag{17}$$

By (11), the inequality (17) reduces to

$$\begin{aligned}
\frac{\Delta f}{f} &\le \frac{n^2}{4n_2}||H||^2 + \frac{c-3u^2}{4}n_1 - \left(\frac{c+u^2}{4}+u'\right) + \frac{3(c+u^2)}{4n_2}\sum_{\substack{1\le j\le n_1\\ n_1+1\le t\le n}} g^2(Pe_j,e_t)\\
&\le \frac{n^2}{4n_2}||H||^2 + \frac{c-3u^2}{4}n_1 - \left(\frac{c+u^2}{4}+u'\right) + \frac{3(c+u^2)}{4}\min\left\{\frac{n_1}{n_2},1\right\}.
\end{aligned} \tag{18}$$

We distinguish two cases:

(a) $n_1 \le n_2$, in this case the inequality (18) implies (6).

(b) $n_1 > n_2$, in this case (18) also becomes (6). It completes the proof. $\boxed{QED}$

4 Corollary. *Let $x : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of an n-dimensional totally real warped product into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. Then, we have*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}||H||^2 + \frac{c-3u^2}{4}n_1 - \frac{c+u^2}{4} - u', \tag{19}$$

where, $n_i = dimM_i, i = 1, 2$, and Δ is the Laplacian operator of M_1.

Moreover, the equality case of (19) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where, $H_i, i = 1, 2$ are the partial mean curvatures.

PROOF. Let $M_1 \times_f M_2$ be a totally real warped product into $\tilde{M}(c)$. Then we have $g(Pe_i, e_s) = 0$ for $0 \le i \le n_1$, $n_1 + 1 \le s \le n$. Therefore, by (18) we can easily obtain the inequality (19). Also, we see that the equality sign of (18) holds if and only if

$$h^r_{jt} = 0, \quad 1 \le j \le n_1, \quad n_1 + 1 \le t \le n, \quad n + 1 \le r \le 2m + 1, \tag{20}$$

and

$$\sum_{i=1}^{n_1} h^r_{ii} = \sum_{t=n_1+1}^{n} h^r_{tt} = 0, \quad n + 2 \le r \le 2m + 1. \tag{21}$$

Obviously (20) is equivalent to the mixed totally geodesic of the warped product $M_1 \times_f M_2$ and (15) and (21) imply $n_1H_1 = n_2H_2$. The converse statement is straightforward. $\boxed{QED}$

5 Corollary. *Let $M_1 \times_f M_2$ be a totally real warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose the structure vector ξ is tangent to M_1 and a warping function f is a harmonic. Then, $M_1 \times_f M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ with $c < \frac{1}{n_1-1}(u^2 + 3n_1u^2 + 4u')$.*

6 Corollary. *Let $M_1 \times_f M_2$ be a totally real warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose the structure vector ξ is tangent to M_1. If the warping function f of $M_1 \times_f M_2$ is an eigenfunction of the Laplacian on M_1 with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ dose not admit a minimal totally real immersion into a locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c with $c \le \frac{1}{n_1-1}(u^2 + 3n_1u^2 + 4u')$.*

7 Corollary. *Let $M_1 \times_f M_2$ be a compact minimal totally real warped product in a locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c such that the structure vector ξ is tangent to M_1 and $c \le \frac{1}{n_1-1}(u^2 + 3n_1u^2 + 4u')$. Then $M_1 \times_f M_2$ is a Riemannian product.*

8 Theorem. *Let $x : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of an n-dimensional warped product into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. Then, we have*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}||H||^2 + \frac{c - 3u^2}{4}n_1 - \left(\frac{c + u^2}{4} + u'\right)\frac{n_1}{n_2} + \frac{3(c + u^2)}{4},$$

where $n_i = dimM_i, i = 1, 2$, and Δ is the Laplacian operator of M_1.

9 Corollary. *Let $x : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of an n-dimensional totally real warped product into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. Then, we have*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}||H||^2 + \frac{c-3u^2}{4}n_1 - \left(\frac{c+u^2}{4} + u'\right)\frac{n_1}{n_2}, \tag{22}$$

where, $n_i = dimM_i, i = 1, 2$, and Δ is the Laplacian operator of M_1.

Moreover, the equality case of (22) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where, $H_i, i = 1, 2$ are the partial mean curvatures.

10 Corollary. *Let $M_1 \times_f M_2$ be a totally real warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose the structure vector ξ is tangent to M_2 and a warping function f is a harmonic. Then, $M_1 \times_f M_2$ admits no minimal totally real immersion into a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ with $c < \frac{1}{n_2-1}(u^2 + 3n_2u^2 + 4u')$.*

11 Corollary. *Let $M_1 \times_f M_2$ be a totally real warped product in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c whose the structure vector ξ is tangent to M_2. If the warping function f of $M_1 \times_f M_2$ is an eigenfunction of the Laplacian on M_1 with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ dose not admit a minimal totally real immersion into a locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c with $c \leq \frac{1}{n_2-1}(u^2 + 3n_2u^2 + 4u')$.*

12 Corollary. *Let $M_1 \times_f M_2$ be a compact minimal totally real warped product in a locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c such that the structure vector ξ is tangent to M_2 and $c \leq \frac{1}{n_2-1}(u^2 + 3n_2u^2 + 4u')$. Then $M_1 \times_f M_2$ is a Riemannian product.*

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